Problem 4.1

(a) Work out all of the **canonical commutation relations** for components of the operators **r** and **p**: $[x, y]$, $[x, p_y]$, $[x, p_x]$, $[p_y, p_z]$, and so on. *Answer:*

$$[r_i, p_j] = -[p_i, r_j] = i\hbar\delta_{ij}, \quad [r_i, r_j] = [p_i, p_j] = 0, \tag{4.10}$$

where the indices stand for x, y, or z, and $r_x = x$, $r_y = y$, and $r_z = z$.

(b) Confirm the three-dimensional version of **Ehrenfest's theorem,**

$$\frac{d}{dt}\langle \mathbf{r} \rangle = \frac{1}{m}\langle \mathbf{p} \rangle, \quad \text{and} \quad \frac{d}{dt}\langle \mathbf{p} \rangle = \langle -\nabla V \rangle. \tag{4.11}$$

(Each of these, of course, stands for *three* equations—one for each component.) *Hint:* First check that the "generalized" Ehrenfest theorem, Equation 3.73, is valid in three dimensions.

(c) Formulate **Heisenberg's uncertainty principle** in three dimensions. *Answer:*

$$\sigma_x \sigma_{p_x} \geq \hbar/2, \quad \sigma_y \sigma_{p_y} \geq \hbar/2, \quad \sigma_z \sigma_{p_z} \geq \hbar/2, \tag{4.12}$$

but there is no restriction on, say, $\sigma_x \sigma_{p_y}$.

[TYPO: This is bad notation because i represents both $\sqrt{-1}$ and an index.]

Solution

Part (a)

Evaluate the commutation relations, using the test function $f = f(x, y, z)$. The mixed partial derivatives of f are equal by Clairaut's theorem.

$$[\hat{x}, \hat{y}]\, f = (\hat{x}\hat{y} - \hat{y}\hat{x})f = (xy - yx)f = 0$$

$$[\hat{y}, \hat{z}]\, f = (\hat{y}\hat{z} - \hat{z}\hat{y})f = (yz - zy)f = 0$$

$$[\hat{x}, \hat{z}]\, f = (\hat{x}\hat{z} - \hat{z}\hat{x})f = (xz - zx)f = 0$$

$$\begin{aligned}
[\hat{p}_x, \hat{p}_y]\, f = (\hat{p}_x\hat{p}_y - \hat{p}_y\hat{p}_x)f &= \left[\left(-i\hbar\frac{\partial}{\partial x}\right)\left(-i\hbar\frac{\partial}{\partial y}\right) - \left(-i\hbar\frac{\partial}{\partial y}\right)\left(-i\hbar\frac{\partial}{\partial x}\right)\right] f \\
&= \left(-\hbar^2\frac{\partial^2}{\partial x \partial y} + \hbar^2\frac{\partial^2}{\partial y \partial x}\right) f \\
&= -\hbar^2\frac{\partial^2 f}{\partial x \partial y} + \hbar^2\frac{\partial^2 f}{\partial y \partial x} \\
&= 0
\end{aligned}$$

Evaluate more of the commutation relations.

$$
\begin{aligned}
[\hat{p}_y, \hat{p}_z]\, f = (\hat{p}_y\hat{p}_z - \hat{p}_z\hat{p}_y) f &= \left[\left(-i\hbar\frac{\partial}{\partial y}\right)\left(-i\hbar\frac{\partial}{\partial z}\right) - \left(-i\hbar\frac{\partial}{\partial z}\right)\left(-i\hbar\frac{\partial}{\partial y}\right)\right] f \\
&= \left(-\hbar^2\frac{\partial^2}{\partial y\partial z} + \hbar^2\frac{\partial^2}{\partial z\partial y}\right) f \\
&= -\hbar^2\frac{\partial^2 f}{\partial y\partial z} + \hbar^2\frac{\partial^2 f}{\partial z\partial y} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
[\hat{p}_x, \hat{p}_z]\, f = (\hat{p}_x\hat{p}_z - \hat{p}_z\hat{p}_x) f &= \left[\left(-i\hbar\frac{\partial}{\partial x}\right)\left(-i\hbar\frac{\partial}{\partial z}\right) - \left(-i\hbar\frac{\partial}{\partial z}\right)\left(-i\hbar\frac{\partial}{\partial x}\right)\right] f \\
&= \left(-\hbar^2\frac{\partial^2}{\partial x\partial z} + \hbar^2\frac{\partial^2}{\partial z\partial x}\right) f \\
&= -\hbar^2\frac{\partial^2 f}{\partial x\partial z} + \hbar^2\frac{\partial^2 f}{\partial z\partial x} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
[\hat{x}, \hat{p}_x]\, f = (\hat{x}\hat{p}_x - \hat{p}_x\hat{x}) f &= \left[x\left(-i\hbar\frac{\partial}{\partial x}\right) - \left(-i\hbar\frac{\partial}{\partial x}\right)x\right] f \\
&= -i\hbar x\frac{\partial}{\partial x}(f) + i\hbar\frac{\partial}{\partial x}(xf) \\
&= -i\hbar x\frac{\partial f}{\partial x} + i\hbar x\frac{\partial f}{\partial x} + i\hbar f \\
&= i\hbar f
\end{aligned}
$$

$$
\begin{aligned}
[\hat{x}, \hat{p}_y]\, f = (\hat{x}\hat{p}_y - \hat{p}_y\hat{x}) f &= \left[x\left(-i\hbar\frac{\partial}{\partial y}\right) - \left(-i\hbar\frac{\partial}{\partial y}\right)x\right] f \\
&= -i\hbar x\frac{\partial}{\partial y}(f) + i\hbar\frac{\partial}{\partial y}(xf) \\
&= -i\hbar x\frac{\partial f}{\partial y} + i\hbar x\frac{\partial f}{\partial y} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
[\hat{x}, \hat{p}_z]\, f = (\hat{x}\hat{p}_z - \hat{p}_z\hat{x}) f &= \left[x\left(-i\hbar\frac{\partial}{\partial z}\right) - \left(-i\hbar\frac{\partial}{\partial z}\right)x\right] f \\
&= -i\hbar x\frac{\partial}{\partial z}(f) + i\hbar\frac{\partial}{\partial z}(xf) \\
&= -i\hbar x\frac{\partial f}{\partial z} + i\hbar x\frac{\partial f}{\partial z} \\
&= 0
\end{aligned}
$$

Evaluate more of the commutation relations.

$$\begin{aligned}
[\hat{y}, \hat{p}_x]\, f = (\hat{y}\hat{p}_x - \hat{p}_x\hat{y})f &= \left[y\left(-i\hbar\frac{\partial}{\partial x}\right) - \left(-i\hbar\frac{\partial}{\partial x}\right)y\right] f \\
&= -i\hbar y\frac{\partial}{\partial x}(f) + i\hbar\frac{\partial}{\partial x}(yf) \\
&= -i\hbar y\frac{\partial f}{\partial x} + i\hbar y\frac{\partial f}{\partial x} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
[\hat{y}, \hat{p}_y]\, f = (\hat{y}\hat{p}_y - \hat{p}_y\hat{y})f &= \left[y\left(-i\hbar\frac{\partial}{\partial y}\right) - \left(-i\hbar\frac{\partial}{\partial y}\right)y\right] f \\
&= -i\hbar y\frac{\partial}{\partial y}(f) + i\hbar\frac{\partial}{\partial y}(yf) \\
&= -i\hbar y\frac{\partial f}{\partial y} + i\hbar y\frac{\partial f}{\partial y} + i\hbar f \\
&= i\hbar f
\end{aligned}$$

$$\begin{aligned}
[\hat{y}, \hat{p}_z]\, f = (\hat{y}\hat{p}_z - \hat{p}_z\hat{y})f &= \left[y\left(-i\hbar\frac{\partial}{\partial z}\right) - \left(-i\hbar\frac{\partial}{\partial z}\right)y\right] f \\
&= -i\hbar y\frac{\partial}{\partial z}(f) + i\hbar\frac{\partial}{\partial z}(yf) \\
&= -i\hbar y\frac{\partial f}{\partial z} + i\hbar y\frac{\partial f}{\partial z} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
[\hat{z}, \hat{p}_x]\, f = (\hat{z}\hat{p}_x - \hat{p}_x\hat{z})f &= \left[z\left(-i\hbar\frac{\partial}{\partial x}\right) - \left(-i\hbar\frac{\partial}{\partial x}\right)z\right] f \\
&= -i\hbar z\frac{\partial}{\partial x}(f) + i\hbar\frac{\partial}{\partial x}(zf) \\
&= -i\hbar z\frac{\partial f}{\partial x} + i\hbar z\frac{\partial f}{\partial x} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
[\hat{z}, \hat{p}_y]\, f = (\hat{z}\hat{p}_y - \hat{p}_y\hat{z})f &= \left[z\left(-i\hbar\frac{\partial}{\partial y}\right) - \left(-i\hbar\frac{\partial}{\partial y}\right)z\right] f \\
&= -i\hbar z\frac{\partial}{\partial y}(f) + i\hbar\frac{\partial}{\partial y}(zf) \\
&= -i\hbar z\frac{\partial f}{\partial y} + i\hbar z\frac{\partial f}{\partial y} \\
&= 0
\end{aligned}$$

Evaluate the last commutation relation.

$$\begin{aligned}[\hat{z}, \hat{p}_z] f = (\hat{z}\hat{p}_z - \hat{p}_z\hat{z})f &= \left[z\left(-i\hbar\frac{\partial}{\partial z}\right) - \left(-i\hbar\frac{\partial}{\partial z}\right)z\right]f \\ &= -i\hbar z\frac{\partial}{\partial z}(f) + i\hbar\frac{\partial}{\partial z}(zf) \\ &= -i\hbar z\frac{\partial f}{\partial z} + i\hbar z\frac{\partial f}{\partial z} + i\hbar f \\ &= i\hbar f\end{aligned}$$

Therefore,

$$\begin{aligned}
[\hat{x}, \hat{y}] &= 0 & &\Rightarrow & [\hat{y}, \hat{x}] &= -[\hat{x}, \hat{y}] = 0 \\
[\hat{y}, \hat{z}] &= 0 & &\Rightarrow & [\hat{z}, \hat{y}] &= -[\hat{y}, \hat{z}] = 0 \\
[\hat{x}, \hat{z}] &= 0 & &\Rightarrow & [\hat{z}, \hat{x}] &= -[\hat{x}, \hat{z}] = 0 \\
[\hat{p}_x, \hat{p}_y] &= 0 & &\Rightarrow & [\hat{p}_y, \hat{p}_x] &= -[\hat{p}_x, \hat{p}_y] = 0 \\
[\hat{p}_y, \hat{p}_z] &= 0 & &\Rightarrow & [\hat{p}_z, \hat{p}_y] &= -[\hat{p}_y, \hat{p}_z] = 0 \\
[\hat{p}_x, \hat{p}_z] &= 0 & &\Rightarrow & [\hat{p}_z, \hat{p}_x] &= -[\hat{p}_x, \hat{p}_z] = 0 \\
[\hat{x}, \hat{p}_x] &= i\hbar & &\Rightarrow & [\hat{p}_x, \hat{x}] &= -[\hat{x}, \hat{p}_x] = -i\hbar \\
[\hat{x}, \hat{p}_y] &= 0 & &\Rightarrow & [\hat{p}_y, \hat{x}] &= -[\hat{x}, \hat{p}_y] = 0 \\
[\hat{x}, \hat{p}_z] &= 0 & &\Rightarrow & [\hat{p}_z, \hat{x}] &= -[\hat{x}, \hat{p}_z] = 0 \\
[\hat{y}, \hat{p}_x] &= 0 & &\Rightarrow & [\hat{p}_x, \hat{y}] &= -[\hat{y}, \hat{p}_x] = 0 \\
[\hat{y}, \hat{p}_y] &= i\hbar & &\Rightarrow & [\hat{p}_y, \hat{y}] &= -[\hat{y}, \hat{p}_y] = -i\hbar \\
[\hat{y}, \hat{p}_z] &= 0 & &\Rightarrow & [\hat{p}_z, \hat{y}] &= -[\hat{y}, \hat{p}_z] = 0 \\
[\hat{z}, \hat{p}_x] &= 0 & &\Rightarrow & [\hat{p}_x, \hat{z}] &= -[\hat{z}, \hat{p}_x] = 0 \\
[\hat{z}, \hat{p}_y] &= 0 & &\Rightarrow & [\hat{p}_y, \hat{z}] &= -[\hat{z}, \hat{p}_y] = 0 \\
[\hat{z}, \hat{p}_z] &= i\hbar & &\Rightarrow & [\hat{p}_z, \hat{z}] &= -[\hat{z}, \hat{p}_z] = -i\hbar.
\end{aligned}$$

Part (b)

The governing equation for the wave function in three dimensions is

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V(x, y, z, t)\Psi(x, y, z, t).$$

Divide both sides by $i\hbar$.

$$\frac{\partial\Psi}{\partial t} = \frac{i\hbar}{2m}\nabla^2\Psi - \frac{i}{\hbar}V\Psi$$

Take the complex conjugate of both sides.

$$\frac{\partial\Psi^*}{\partial t} = -\frac{i\hbar}{2m}\nabla^2\Psi^* + \frac{i}{\hbar}V\Psi^*$$

Calculate the time-derivative of the expectation value of x_j, where x_j is x, y, or z depending if j is 1, 2, or 3, respectively.

$$\begin{aligned}
\frac{d}{dt}\langle x_j\rangle &= \frac{d}{dt}\langle\Psi\,|\,\hat{x}_j\,|\,\Psi\rangle \\
&= \frac{d}{dt}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi^*(x,y,z,t)(x_j)\Psi(x,y,z,t)\,dx\,dy\,dz \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial}{\partial t}\left[\Psi^*(x,y,z,t)(x_j)\Psi(x,y,z,t)\right]dx\,dy\,dz \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_j\left(\frac{\partial\Psi^*}{\partial t}\Psi+\Psi^*\frac{\partial\Psi}{\partial t}\right)dx\,dy\,dz \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_j\left[\left(-\frac{i\hbar}{2m}\nabla^2\Psi^*+\frac{i}{\hbar}V\Psi^*\right)\Psi+\Psi^*\left(\frac{i\hbar}{2m}\nabla^2\Psi-\frac{i}{\hbar}V\Psi\right)\right]dx\,dy\,dz \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_j\left(-\frac{i\hbar}{2m}\Psi\nabla^2\Psi^*+\frac{i\hbar}{2m}\Psi^*\nabla^2\Psi\right)dx\,dy\,dz \\
&= \frac{i\hbar}{2m}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_j\left(\Psi^*\nabla^2\Psi-\Psi\nabla^2\Psi^*\right)dx\,dy\,dz \\
&= \frac{i\hbar}{2m}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_j\Psi^*\nabla^2\Psi\,dx\,dy\,dz-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_j\Psi\nabla^2\Psi^*\,dx\,dy\,dz\right) \qquad (1)
\end{aligned}$$

Green's first identity is essentially the multidimensional integration-by-parts formula.

$$\iiint\limits_D v\nabla^2u\,d\mathcal{V}=\oiint\limits_{\text{bdy }D}v\frac{\partial u}{\partial n}\,d\mathcal{S}-\iiint\limits_D\nabla v\cdot\nabla u\,d\mathcal{V}$$

u and v are any two differentiable functions, D is the domain that these two functions are defined on, bdy D is the boundary of D, and

$$\frac{\partial u}{\partial n}=\hat{\mathbf{n}}\cdot\nabla u$$

is the directional derivative of u in the outward normal direction from the boundary. For the infinite domain here, the boundary term vanishes, as Ψ and its derivatives vanish as $|\mathbf{x}|\to\infty$. Using Green's first identity, the two integrals in equation (1) become

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_j\Psi^*\nabla^2\Psi\,dx\,dy\,dz=0-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\nabla(x_j\Psi^*)\cdot\nabla\Psi\,dx\,dy\,dz$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_j\Psi\nabla^2\Psi^*\,dx\,dy\,dz=0-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\nabla(x_j\Psi)\cdot\nabla\Psi^*\,dx\,dy\,dz,$$

which means

$$\begin{aligned}
\frac{d}{dt}\langle x_j\rangle &= \frac{i\hbar}{2m}\left[-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\nabla(x_j\Psi^*)\cdot\nabla\Psi\,dx\,dy\,dz+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\nabla(x_j\Psi)\cdot\nabla\Psi^*\,dx\,dy\,dz\right] \\
&= \frac{i\hbar}{2m}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\nabla(x_j\Psi)\cdot\nabla\Psi^*-\nabla(x_j\Psi^*)\cdot\nabla\Psi\right]dx\,dy\,dz.
\end{aligned}$$

Simplify the integrand.

$$\begin{aligned}
\frac{d}{dt}\langle x_j\rangle &= \frac{i\hbar}{2m}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\nabla(x_j\Psi)\cdot\nabla\Psi^* - \nabla(x_j\Psi^*)\cdot\nabla\Psi\right]d\mathcal{V} \\
&= \frac{i\hbar}{2m}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k\frac{\partial}{\partial x_k}\right)(x_j\Psi)\cdot\left(\sum_{l=1}^{3}\boldsymbol{\delta}_l\frac{\partial}{\partial x_l}\right)\Psi^* - \left(\sum_{k=1}^{3}\boldsymbol{\delta}_k\frac{\partial}{\partial x_k}\right)(x_j\Psi^*)\cdot\left(\sum_{l=1}^{3}\boldsymbol{\delta}_l\frac{\partial}{\partial x_l}\right)\Psi\right]d\mathcal{V} \\
&= \frac{i\hbar}{2m}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\sum_{k=1}^{3}\boldsymbol{\delta}_k\frac{\partial}{\partial x_k}(x_j\Psi)\cdot\sum_{l=1}^{3}\boldsymbol{\delta}_l\frac{\partial\Psi^*}{\partial x_l} - \sum_{k=1}^{3}\boldsymbol{\delta}_k\frac{\partial}{\partial x_k}(x_j\Psi^*)\cdot\sum_{l=1}^{3}\boldsymbol{\delta}_l\frac{\partial\Psi}{\partial x_l}\right]d\mathcal{V} \\
&= \frac{i\hbar}{2m}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_k\cdot\boldsymbol{\delta}_l)\frac{\partial}{\partial x_k}(x_j\Psi)\frac{\partial\Psi^*}{\partial x_l} - \sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_k\cdot\boldsymbol{\delta}_l)\frac{\partial}{\partial x_k}(x_j\Psi^*)\frac{\partial\Psi}{\partial x_l}\right]d\mathcal{V} \\
&= \frac{i\hbar}{2m}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\sum_{k=1}^{3}\sum_{l=1}^{3}\delta_{kl}\frac{\partial}{\partial x_k}(x_j\Psi)\frac{\partial\Psi^*}{\partial x_l} - \sum_{k=1}^{3}\sum_{l=1}^{3}\delta_{kl}\frac{\partial}{\partial x_k}(x_j\Psi^*)\frac{\partial\Psi}{\partial x_l}\right]d\mathcal{V} \\
&= \frac{i\hbar}{2m}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\sum_{k=1}^{3}\frac{\partial}{\partial x_k}(x_j\Psi)\frac{\partial\Psi^*}{\partial x_k} - \sum_{k=1}^{3}\frac{\partial}{\partial x_k}(x_j\Psi^*)\frac{\partial\Psi}{\partial x_k}\right]d\mathcal{V} \\
&= \frac{i\hbar}{2m}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\sum_{k=1}^{3}\left(\frac{\partial x_j}{\partial x_k}\Psi + x_j\frac{\partial\Psi}{\partial x_k}\right)\frac{\partial\Psi^*}{\partial x_k} - \sum_{k=1}^{3}\left(\frac{\partial x_j}{\partial x_k}\Psi^* + x_j\frac{\partial\Psi^*}{\partial x_k}\right)\frac{\partial\Psi}{\partial x_k}\right]d\mathcal{V} \\
&= \frac{i\hbar}{2m}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\sum_{k=1}^{3}\left(\delta_{jk}\Psi + x_j\frac{\partial\Psi}{\partial x_k}\right)\frac{\partial\Psi^*}{\partial x_k} - \sum_{k=1}^{3}\left(\delta_{jk}\Psi^* + x_j\frac{\partial\Psi^*}{\partial x_k}\right)\frac{\partial\Psi}{\partial x_k}\right]d\mathcal{V} \\
&= \frac{i\hbar}{2m}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\sum_{k=1}^{3}\delta_{jk}\Psi\frac{\partial\Psi^*}{\partial x_k} + \cancel{\sum_{k=1}^{3}x_j\frac{\partial\Psi}{\partial x_k}\frac{\partial\Psi^*}{\partial x_k}} - \sum_{k=1}^{3}\delta_{jk}\Psi^*\frac{\partial\Psi}{\partial x_k} - \cancel{\sum_{k=1}^{3}x_j\frac{\partial\Psi^*}{\partial x_k}\frac{\partial\Psi}{\partial x_k}}\right)d\mathcal{V} \\
&= \frac{i\hbar}{2m}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\Psi\frac{\partial\Psi^*}{\partial x_j} - \Psi^*\frac{\partial\Psi}{\partial x_j}\right)d\mathcal{V} \\
&= -\frac{i\hbar}{2m}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\Psi^*\frac{\partial\Psi}{\partial x_j} - \Psi\frac{\partial\Psi^*}{\partial x_j}\right)d\mathcal{V} \\
&= -\frac{i\hbar}{2m}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\left(\Psi^*\frac{\partial\Psi}{\partial x_j} - \Psi\frac{\partial\Psi^*}{\partial x_j}\right)dx_j\right]d\mathcal{A} \\
&= -\frac{i\hbar}{2m}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\Psi^*\frac{\partial\Psi}{\partial x_j}\,dx_j - \int_{-\infty}^{\infty}\Psi\frac{\partial\Psi^*}{\partial x_j}\,dx_j\right)d\mathcal{A} \\
&= -\frac{i\hbar}{2m}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\Psi^*\frac{\partial\Psi}{\partial x_j}\,dx_j - \left(\underbrace{\Psi\Psi^*\Big|_{-\infty}^{\infty}}_{=\,0} - \int_{-\infty}^{\infty}\frac{\partial\Psi}{\partial x_j}\Psi^*\,dx_j\right)\right]d\mathcal{A} \\
&= -\frac{i\hbar}{2m}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\Psi^*\frac{\partial\Psi}{\partial x_j}\,dx_j + \int_{-\infty}^{\infty}\Psi^*\frac{\partial\Psi}{\partial x_j}\,dx_j\right]d\mathcal{A}
\end{aligned}$$

Therefore, in component form,

$$\begin{aligned}
\frac{d}{dt}\langle x_j\rangle &= -\frac{i\hbar}{2m}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(2\int_{-\infty}^{\infty}\Psi^*\frac{\partial\Psi}{\partial x_j}\,dx_j\right)d\mathcal{A} \\
&= -\frac{i\hbar}{m}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi^*\frac{\partial\Psi}{\partial x_j}\,dx\,dy\,dz \\
&= \frac{1}{m}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi^*\left(-i\hbar\frac{\partial}{\partial x_j}\right)\Psi\,dx\,dy\,dz \\
&= \frac{1}{m}\langle\Psi\,|\,\hat{p}_j\,|\,\Psi\rangle \\
&= \frac{1}{m}\langle p_j\rangle,
\end{aligned}$$

or as a vector equation,

$$\frac{d}{dt}\langle\mathbf{r}\rangle = \frac{1}{m}\langle\mathbf{p}\rangle.$$

Now take the time-derivative of the expectation value of momentum.

$$\begin{aligned}
\frac{d}{dt}\langle\mathbf{p}\rangle &= \frac{d}{dt}\langle\Psi\,|\,\hat{p}\,|\,\Psi\rangle \\
&= \frac{d}{dt}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi^*(x,y,z,t)\,(-i\hbar\nabla)\,\Psi(x,y,z,t)\,dx\,dy\,dz \\
&= -i\hbar\frac{d}{dt}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi^*(x,y,z,t)\nabla\Psi(x,y,z,t)\,dx\,dy\,dz \\
&= -i\hbar\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial}{\partial t}\left[\Psi^*(x,y,z,t)\nabla\Psi(x,y,z,t)\right]dx\,dy\,dz \\
&= -i\hbar\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\left(\frac{\partial\Psi^*}{\partial t}\right)\nabla\Psi+\Psi^*\nabla\left(\frac{\partial\Psi}{\partial t}\right)\right]dx\,dy\,dz \\
&= -i\hbar\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\left(-\frac{i\hbar}{2m}\nabla^2\Psi^*+\frac{i}{\hbar}V\Psi^*\right)\nabla\Psi+\Psi^*\nabla\left(\frac{i\hbar}{2m}\nabla^2\Psi-\frac{i}{\hbar}V\Psi\right)\right]dx\,dy\,dz \\
&= -i\hbar\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[-\frac{i\hbar}{2m}\nabla^2\Psi^*\nabla\Psi+\frac{i}{\hbar}V\Psi^*\nabla\Psi+\frac{i\hbar}{2m}\Psi^*\nabla(\nabla^2\Psi)-\frac{i}{\hbar}\Psi^*\nabla(V\Psi)\right]dx\,dy\,dz \\
&= -i\hbar\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[-\frac{i\hbar}{2m}\nabla^2\Psi^*\nabla\Psi+\cancel{\frac{i}{\hbar}V\Psi^*\nabla\Psi}+\frac{i\hbar}{2m}\Psi^*\nabla(\nabla^2\Psi)-\frac{i}{\hbar}\Psi^*\nabla V\Psi-\cancel{\frac{i}{\hbar}\Psi^*V\nabla\Psi}\right]dx\,dy\,dz \\
&= -i\hbar\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[-\frac{i\hbar}{2m}\nabla^2\Psi^*\nabla\Psi+\frac{i\hbar}{2m}\Psi^*\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}\right)\left(\sum_{k=1}^{3}\frac{\partial^2\Psi}{\partial x_k^2}\right)-\frac{i}{\hbar}\Psi^*\nabla V\Psi\right]dx\,dy\,dz \\
&= -i\hbar\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[-\frac{i\hbar}{2m}\nabla^2\Psi^*\nabla\Psi+\frac{i\hbar}{2m}\Psi^*\sum_{k=1}^{3}\frac{\partial^2}{\partial x_k^2}\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial\Psi}{\partial x_j}\right)-\frac{i}{\hbar}\Psi^*\nabla V\Psi\right]dx\,dy\,dz
\end{aligned}$$

As a result,

$$\begin{aligned}
\frac{d}{dt}\langle \mathbf{p}\rangle &= -i\hbar\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[-\frac{i\hbar}{2m}\nabla^2\Psi^*\nabla\Psi + \frac{i\hbar}{2m}\Psi^*\nabla^2(\nabla\Psi) - \frac{i}{\hbar}\Psi^*\nabla V\Psi\right]dx\,dy\,dz \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[-\frac{\hbar^2}{2m}\nabla^2\Psi^*\nabla\Psi + \frac{\hbar^2}{2m}\Psi^*\nabla^2(\nabla\Psi) - \Psi^*\nabla V\Psi\right]dx\,dy\,dz \\
&= \frac{\hbar^2}{2m}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\Psi^*\nabla^2(\nabla\Psi) - \nabla^2\Psi^*\nabla\Psi]\,dx\,dy\,dz - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi^*\nabla V\Psi\,dx\,dy\,dz.
\end{aligned}$$

Green's second identity says that

$$\iiint\limits_{D}\left(u\nabla^2 v - v\nabla^2 u\right)d\mathcal{V} = \oiint\limits_{\text{bdy } D}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)d\mathcal{S}$$

for any two functions, u and v, on a domain D. For $u = \Psi^*$ and $v = \nabla\Psi$ and D being all of space, Green's second identity becomes

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\Psi^*\nabla^2(\nabla\Psi) - \nabla^2\Psi^*\nabla\Psi]\,d\mathcal{V} = 0$$

because Ψ and its derivatives go to zero as $|\mathbf{x}| \to \infty$. Therefore,

$$\begin{aligned}
\frac{d}{dt}\langle \mathbf{p}\rangle &= \frac{\hbar^2}{2m}(0) - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi^*\nabla V\Psi\,dx\,dy\,dz \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi^*(-\nabla V)\Psi\,dx\,dy\,dz \\
&= \langle \Psi \mid -\nabla V \mid \Psi\rangle \\
&= \langle -\nabla V\rangle.
\end{aligned}$$

Part (c)

Apply the generalized uncertainty principle for two observables,

$$\sigma_A^2\sigma_B^2 \geq \left(\frac{1}{2i}\left\langle\left[\hat{A},\hat{B}\right]\right\rangle\right)^2,$$

to the results of part (a).

$$\begin{aligned}
[\hat{x},\hat{y}] &= 0 & &\Rightarrow & &\sigma_x\sigma_y \geq 0 \\
[\hat{y},\hat{z}] &= 0 & &\Rightarrow & &\sigma_y\sigma_z \geq 0 \\
[\hat{x},\hat{z}] &= 0 & &\Rightarrow & &\sigma_x\sigma_z \geq 0 \\
[\hat{p}_x,\hat{p}_y] &= 0 & &\Rightarrow & &\sigma_{p_x}\sigma_{p_y} \geq 0 \\
[\hat{p}_y,\hat{p}_z] &= 0 & &\Rightarrow & &\sigma_{p_y}\sigma_{p_z} \geq 0 \\
[\hat{p}_x,\hat{p}_z] &= 0 & &\Rightarrow & &\sigma_{p_x}\sigma_{p_z} \geq 0 \\
[\hat{x},\hat{p}_x] &= i\hbar & &\Rightarrow & &\sigma_x\sigma_{p_x} \geq \frac{\hbar}{2} \\
[\hat{x},\hat{p}_y] &= 0 & &\Rightarrow & &\sigma_x\sigma_{p_y} \geq 0 \\
[\hat{x},\hat{p}_z] &= 0 & &\Rightarrow & &\sigma_x\sigma_{p_z} \geq 0 \\
[\hat{y},\hat{p}_x] &= 0 & &\Rightarrow & &\sigma_y\sigma_{p_x} \geq 0 \\
[\hat{y},\hat{p}_y] &= i\hbar & &\Rightarrow & &\sigma_y\sigma_{p_y} \geq \frac{\hbar}{2} \\
[\hat{y},\hat{p}_z] &= 0 & &\Rightarrow & &\sigma_y\sigma_{p_z} \geq 0 \\
[\hat{z},\hat{p}_x] &= 0 & &\Rightarrow & &\sigma_z\sigma_{p_x} \geq 0 \\
[\hat{z},\hat{p}_y] &= 0 & &\Rightarrow & &\sigma_z\sigma_{p_y} \geq 0 \\
[\hat{z},\hat{p}_z] &= i\hbar & &\Rightarrow & &\sigma_z\sigma_{p_z} \geq \frac{\hbar}{2}
\end{aligned}$$